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the placenta. The two sexes are nearly equally apt to be affected—of 243 affected persons 56.8% are males. The disease tends to recur without a break in the generations and is equally apt to come down the male and the female line. Consequently it looks as though the hereditary factor in neurofibromatosis is a dominant one. In each affected fraternity, indeed, about 50% of the individuals are affected, as is expected if it is a dominant trait. Actually 43.5% were found affected. In some cases, however, a generation is skipped—a result that can be explained on the hypothesis of occasional failure of dominance.

The symptoms of neurofibromatosis are very diverse. But inside of one family they are apt to be alike. This speaks strongly for the hypothesis of an inheritance factor. Similarly the location of the principal tumors is apt to be the same in one family, although it shows the greatest diversity in different families. Other multiple tumors are inherited in the same way as neurofibromata. Thus the tendency to form vascular tumors of the skin and mucous membranes has been shown by Osler (1901) and others since to be a dominant one. Polyadenomata are inherited similarly. Likewise the tendency to form pigmented patches in the skin (ephilides) was shown by Hammer to be a dominant trait. To this same group of heredity belong epidermolysis bullosa, angioneurotic oedema, and persistent hereditary oedema, also such skin diseases as psoriasis, porokeratosis and ichthyosis.

In not a few cases the removal of neurofibromata has been followed by malignant growths, at the same spot. It is plain that neurofibromata are in some way related to cancerous growths. The fact that neurofibromata have an inheritable basis strengthens the view that cancers in general have such a basis.

The complete paper will be published jointly with Dr. S. A. Preiser.

ARITHMETICAL THEORY OF CERTAIN HURWITZIAN CONTINUED FRACTIONS

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The following research is the outcome of the discovery, made some three years ago, that the denominators of the convergents of order $3n$, $3n-2$, and $3n-6$, as well as the numerator of the convergent of order $3n-3$ in the regular continued fraction which represents the base of Naperian logarithms, are all divisible by n. The convergents recur modulo n with a period of $6n$ when n is odd, and with a period of $3n$ when n is even.

To account for these curious theorems, and to place them in their proper setting, it was found necessary to study a more general type of continued fraction first investigated by Hurwitz.[1] The regular continued fraction for the Naperian base was discovered by Roger Cotes[2] to be (2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1,. . . .), or as we may write it $(2, 1, \overline{2n, 1, 1})$, $n = 1, 2, 3,\ldots$ The proof of the remarkable sequence of partial quotients was first given by Euler,[3] by means of the theory of the Riccati equation. Euler established a number of other interesting expansions, such as:

$$\sqrt{e} = (\overline{1, 1, 4n+1}),\ n = 0, 1, 2, 3, \ldots .$$
$$\sqrt[s]{e} = (\overline{1, s\,(2n+1) - 1, 1}),\ n = 0, 1, 2, 3, \ldots .$$
$$\frac{\sqrt[s]{e}+1}{\sqrt[s]{e}-1} = (\overline{(4n+2)\,s}),\ n = 0, 1, 2, 3, \ldots .$$

The continued fractions studied by Hurwitz may be written in the form:

$$(q_1, q_2, q_3, \ldots q_r, \overline{\varphi_1(n), \varphi_2(n), \varphi_3(n), \ldots .\ \varphi_k(n)}),\ n = 0, 1, 2, 3, \ldots .$$

where the 'irregular' partial quotients, $q_1, q_2, \ldots q_r$ are rational, and with the possible exception of q_1, all positive. The functions $\varphi_i(n)$ are rational integral functions whose degrees may, some or all, be zero. If the functions are all of degree zero, the fraction becomes an ordinary periodic continued fraction. The fractions discovered by Euler are seen to be Hurwitzian fractions where all the functions are of degree zero except one which is of the first degree. The general type of such a fraction with no irregular partial quotients is

$$(\overline{a_1, a_2, a_3, \ldots .\ a_{k-1}, ma_k + b}),\ m = 0, 1, 2, \ldots .$$

and for such fractions I have been able to prove the congruences:

$$A_{2nk-1} - 1 \equiv B_{2nk} \equiv 0 \pmod{n},$$
$$A_{2nk} \equiv B_{2nk-1} \equiv (-1)^{nk-1} \pmod{n};$$

Where A_m/B_m is the mth convergent to the continued fraction, and n is any number prime to A_k, A_{k-1} and B_{k-1}; $a_1, a_2, a_3, \ldots a_{k-1}$ being positive or negative integers or zero; A_k, A_{k-1}, B_{k-1} positive or negative integers not zero.

The discussion is based on the following two theorems:

Theorem I. If A_m/B_m is the mth convergent of $(\overline{a_1, a_2, \ldots a_{k-1}, ma_k})$, $m = 1, 2, 3, \ldots$ and A'_m/B'_m is the mth convergent of $(\overline{a_{k-1}, a_{k-2}, \ldots . a_2,}$ $\overline{a_1, -a_k m - 2M})$, $m = 1, 2, 3, \ldots$; where $M = (B_{k-1} + A_{k-2})/A_{k-1}$; where $a_1, a_2, a_3 \ldots a_{k-1}$, are positive or negative integers or zero, and where a_k, A_{k-1} are positive or negative integers, not zero, then:

$$A_{\rho k-1} = (-1)^{\rho-1} A'_{\rho k-1},$$
$$A_{\rho k} = (-1)^{\rho} (A'_{\rho k} + MA'_{\rho k-1})$$

for $\rho = 1, 2, 3, \ldots$.

Theorem II. With A_m/B_m defined as in Theorem I, and A''_m/B''_m defined as the $(m + 1)$st convergent of

$$(K,\ a_{k-1}\ a_{k-2}, \ldots a_2, a_1, L, \overline{a_{k-1}, a_{k-2}, \ldots a_2, a_1, - a_k m - aM}),$$
$$m = 2, 3, 4, 5, \ldots,$$

where $K = (B_{k-1} - A_{k-2}) / A_{k-1}$
and $L = - a_k - 2\ (B^2_{k-1} + B_{k-2}\ A_{k-1}) / A_{k-1}\ B_{k-1}$;

$a_1, a_2 \ldots a_k, A_{k-1}$ and M as before, and B_{k-1} not zero, then:

$$B_{\rho k-1} = (-1)^{\rho-1} A''_{\rho k-1},$$
$$B_{\rho k} = (-1)^{\rho} (A''_{\rho k} + MA''_{\rho k-1})$$

for $\rho = 1, 2, 3, \ldots$.

These two theorems are proved by complete induction. The proof of each proceeds in an unusual and interesting way by assuming both equations of the theorem to hold for $\rho \leqq n$, then proving that the first of the two equations must be valid for $\rho = n + 1$, and then using this result to show that the second holds also for $\rho = n + 1$.

The three fractions considered in these two theorems are then discussed with respect to a modulus n prime to $2a_k$, A_{k-1} and B_{k-1}. It is possible to find an even value of m, say $m = 2\lambda$, which shall be less than $2n$, and which shall satisfy the congruence

$$a_k m + 2M \equiv 0 \pmod{n}$$

This value of m will provide a partial quotient in the second fraction of Theorem I, which shall be congruent to zero modulo n. The two continued fractions of this theorem are then seen to be inverse modulo n as far as this partial quotient. From the properties of inverse continued fractions, we have then $A_{2k\lambda-1} \equiv A'_{2k\lambda-1}$. But by Theorem I itself $A_{2k\lambda-1} = - A'_{2k\lambda-1}$ so that we derive $A_{2k\lambda-1} \equiv 0 \pmod{n}$.

It is possible, using only Theorem I, to show also that $A_{2k\lambda} \equiv B_{2k\lambda-1} \equiv (-1)^{(\lambda(k+1))}$. To prove, however, that $B_{2k\lambda} \equiv M(-1)^{\lambda(k+1)}$ it is necessary to use Theorem II.

Certain interesting palindromic relations develop in the course of the proof, such as

$$A_{2k\lambda-(mk+1)} \equiv (-1)^{(k+1)(\lambda+m)} A_{mk-1} \pmod{n},$$

with a similar congruence in the B's.

With these values for the convergents of order $2k\lambda$ and $2k\lambda - 1$ it is possible to proceed to those of order $2kn$ and $2kn - 1$ with the results as noted above

$$\begin{aligned} A_{2nk} &\equiv B_{2nk-1} \equiv (-1)^{nk-1} \pmod{n}, \\ A_{2nk-1} &\equiv B_{2nk} \equiv 0 \pmod{n}, \end{aligned}$$

and these last results are, without much difficulty, shown to be valid also for the more general fraction

$$(\overline{a_1, a_2, a_3 \ . \ . \ . \ . \ a_{k-1}, a_k m + b}), \ m = 1, 2, 3, \ . \ . \ . \ .$$

where n is prime to a_k, A_{k-1} and B_{k-1}. If we take the still more general fraction with irregular partial quotients:

$$(q_1, q_2, q_3 \ . \ . \ . \ . \ q_r, \overline{a_1, a_2, a_3 \ . \ . \ . \ . \ a_{k-1}, a_k m + b}), \ m = 1, 2, 3, \ . \ . \ .$$

the values given above for the convergents of order $2nk + r$ and $2nk + r - 1$ are no longer correct but must be replaced by the rth and $(r - 1)$st convergents of the fraction $(q_1, q_2, q_3 \ldots\ldots q_r)$.

The above mentioned divisibility theorems concerning the Naperian base are easily derived from these general theorems. The special cases where n is not prime to $2a_k$ and to A_{k-1} and to B_{k-1} have been examined in detail. When A_k is divisible by n the fraction reduces modulo n to a purely periodic fraction of k partial quotients with no relations between them. The number of terms in the period of the convergents to such a fraction depends upon the discriminant D of the quadratic equation of which the fraction is one root. For a prime modulus p the period of the convergents is a divisor of $p + (D/p)$ where (D/p) is the symbol of Legendre.

For the case n a divisor of A_{k-1} (or B_{k-1}) the period of the convergents is a multiple of nke where e is the exponent to which A_k (or B_k) belongs modulo n. It is curious that for these special cases the period of the convergents does not seem to be definitely assignable as in the general case.

In this investigation we are concerned only with the successive values of the numerators and denominators of the convergents, and not at all with the existence or non-existence of a limiting value of the convergents. Certain of the fractions involved are closely related to those called 'semi-regular' whose convergence has been studied by Tietze,[4] and the rules which apply to semi-regular continued fractions may be easily modified to apply to them. The research here carried out is really an arithmetical study of the series of numbers which satisfy certain difference equations and, as Professor Birkhoff has remarked, the theorems are thus related to the lesser Fermat theorem and to Wilson's theorem which are concerned respectively with the numbers which satisfy the difference equations $u_{x+1} = au_x$, and $u_{x+1} = xu_x$.

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[1] Hurwitz, *Zurich, Vierteljahrsch. Natf. Ges.*, **41**, 1896, (34–64).

[2] Cotes, *Phil. Trans., London,* **29**, 1714, (5). (Reference in *Encyc. Sci. Math.*, Paris, Tome I, Vol. I, Fasc. 2, p. 169 note.)

[3] Euler, *Com. Acad. Petropolitanae*, **9**, 1737, (98–137).

[4] Tietze, *Math. Ann., Leipzig*, **70**, 1911, (236–265).

ON CLOSED CURVES DESCRIBED BY A SPHERICAL PENDULUM

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1. The geometrical aspect of the problem of closed curves in a spherical pendulum-motion has apparently never been fully discussed and it is the object of this note to present the results of an investigation of some of the geometric properties of these curves.

The differential equations of the motion are

$$\frac{d^2x}{dt^2} = -S\frac{x}{l}, \quad \frac{d^2y}{dt^2} = -S\frac{y}{l}, \quad \frac{d^2z}{dt^2} = -S\frac{z}{l} - g, \tag{1}$$

in which l is the length of the pendulum, S the variable reaction directed towards the origin (point of suspension). The velocity v of the pendulum-bob is, as is well known, given by $v^2 = h - 2gz$, in which h is a constant depending upon the initial conditions. For z as a function of t, we may obtain without difficulty

$$z = \frac{h}{6g} + \frac{2l^2}{g} p\,(t + w_2). \tag{2}$$

In this Weierstrassian p-function the one half-period w_1 is real; the other w_2 is pure imaginary. Putting $u = t + w_2$, let a and b be the arguments for which z assumes the values $-l$ and $+l$, respectively, and t_1 and t_2 the corresponding complex values of t. It is found that $a = i\,\alpha$, where α is real and positive, and $b = w_1 + i\beta$, with β equal to a fraction of $|w_2|$. The constant of integration may be determined such that $x = r_0$, $y = 0$, when $t = 0$. Under these conditions $\theta = \tan^{-1}(y/x)$ is defined by